When noise decreases deterministic diffusion

Renate Wackerbauer*
Max-Planck-Institute for Physics of Complex Systems, 01187 Dresden, Germany
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Dynamical noise, acting homogeneously in each time step, can enhance the stability of an unstable fixed point. However, if dynamical noise is added locally in state space, additional clear enhancement can be achieved, if this restriction is chosen properly. A systematic analysis of the influence of local and global dynamical noise on the residence time of an unstable state is presented, and optimal parameters for the stabilizing mechanisms are discussed. As a consequence, it is demonstrated that local dynamical noise can yield increased localization in deterministic diffusion models. [S1063-651X(99)08203-3]

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I. INTRODUCTION

The fact that dynamical noise can enhance the stability of an unstable fixed point appears as a somewhat counterintuitive behavior, but is reported in several studies. In the presence of a periodic force, noise-enhanced stability is found in the transient dynamics of an overdamped particle in a noisy cubic potential [1], or in a noisy bistable system operating in a strong forcing regime [2,3]. This is discussed as a noise-induced failure of a switch device in the context of stochastic resonance. However, in the absence of a periodic drive, e.g., in systems which exhibit an unstable fixed point without forcing, dynamical noise can also enhance the stability. In the Lorenz system the switching of a typical trajectory between the two symmetric lobes of the Lorenz attractor can be significantly postponed by dynamical noise [4]. This noise-induced delay of the decay of an unstable state was also reported recently within a model of overdamped Brownian motion in a potential field [5]. Essential mechanisms for such a stabilizing behavior were discussed in detail in the case of one-dimensional discontinuous maps [6], and identified as noise-induced shifts of unstable states resulting in noise-induced attractive regions in state space.

All these studies deal with global dynamical noise, e.g., a deterministic dynamical system is perturbed by a stochastic process homogeneously in each time step. However, if dynamical noise acts only locally in state space, a considerable enhancement of the stability can be achieved, even in cases where there is no enhancement for global noise. For a linear map on a torus this was mentioned briefly in Ref. [6], but not discussed in detail. Thus, in the first part of this paper (Sec. III), the influence of dynamical noise on the decay of an unstable fixed point is investigated, and general conditions are derived to optimize the decrease of instability by local noise.

These findings can be transferred directly to the problem of deterministic diffusion [7–9] (Sec. IV), where chaotic deterministic dynamics induces a diffusive behavior, which is in contrast to ratchetlike devices, where fluctuations can already induce transport [10]. Deterministic diffusion has been studied to explain the dynamics of real physical systems like Josephson junctions in the presence of a microwave field [11,12]. It is also deeply connected with phenomena as quantum dynamical localization [13] and the related Anderson localization [14], which have been used to explain a wide variety of transport or spectroscopic phenomena in the presence of (random) disorder [15]. Increasing attention in deterministic diffusion is also caused by recent advancements of periodic orbit theory [9] and Lévi flight statistics [12]. For example, in Ref. [16], a totally dynamical approach was presented in deriving a Lévi process which demonstrates a link of chaotic dynamical systems and associated random processes. Here we show that local noise in state space can clearly delay deterministic diffusion and therefore enhance localization properties in transport.

II. MODEL

The influence of dynamical noise on the decay of an unstable fixed point at $x=0$ is studied with an antisymmetric one-dimensional map F, $F(-x)=-F(x)$, on the interval $I \in \mathbb{R}$

$$I \to I, \quad x_i \mapsto x_{i+1} = F(x_i + Dg(x_i,\xi_i)), \tag{1}$$

where $g(x_i,\xi_i)$ describes a multiplicative or additive noise term with noise event ξ_i and noise amplitude D. The influence of dynamical noise, restricted to different subsets of the interval I, on the dynamical system F is studied by means of four different noise functions g. In Sec. III A the dynamics is perturbed by additive dynamical noise in the first time step $i=0$, only. Therefore, $g(x_i,\xi_i)=\xi_i\delta_{i,0}$. In Secs. III B and III C, dynamical noise is restricted to a subinterval $[\alpha,\beta]$ and $[-\alpha,-\beta], \alpha,\beta>0$. (By symmetry of the map F, a restriction of noise to $[\alpha,\beta]$ always includes a restriction to $[-\alpha,-\beta]$ throughout this paper, without mentioning it explicitly.) Thus $g(x_i,\xi_i)=\xi_i\Theta(x_i-\alpha)\Theta(\beta-x_i)$, and $\Theta(\,)$ represents the Heaviside function. Finally in Sec. III D global dynamical noise is discussed. That is, the dynamical system is perturbed in each time step i by a random event, and $g(x_i,\xi_i)=\xi_i$.

Two different random processes are considered which can be both experimentally realized, but yield a different en-

*Present address: Department of Chemistry, West Virginia University, Morgantown, WV 26506. Electronic address: wacker@mpipks-dresden.mpg.de hancement of the stability of an unstable fixed point. As random perturbations, *uniform noise* $\xi^{\rm uni}$ as well as *dichotomous Markov noise* ξ^M are considered. Uniform noise is defined as white δ-correlated noise with zero mean ($\langle \xi_i \xi_{i'} \rangle = \delta_{i,i'}$), where ξ_i is uniformly distributed in the interval $\xi_i \in [-1,1]$. For dichotomous Markov noise, ξ_i is uniformly distributed in the set $\xi_i \in \{-1,1\}$. In principle, the results of this paper also hold for Gaussian white noise, not discussed here. This distinction between different types of dynamical noise ($\xi^{\rm uni}, \xi^M$) acting in different subsets of I, is mainly made to probe and optimize specific stabilization mechanisms. For certain cases of g analytical results can be derived. For sake of simplicity, only noise amplitudes $D \leqslant 0.5$ are discussed within this paper.

The uniform noise distribution with amplitude D is given by $h(D\xi_i) = [1/(2D)]\Theta(\xi_i + D)\Theta(D - \xi_i)$, and the corresponding distribution for dichotomous Markov noise is $h(D\xi_i) = \frac{1}{2}(\delta(\xi_i - D) + \delta(\xi_i + D))$. Therefore, a perturbation of a given state x_i by either of the two random processes yields the distribution of perturbed states x_i' as

$$f(x_i', x_i) = \begin{cases} \dfrac{1}{2D}\Theta(x_i' - (x_i - D))\Theta(x_i + D - x_i') & \text{for } \xi^{\rm uni} \\ \dfrac{1}{2}[\delta(x_i' - (x_i - D)) + \delta(x_i' - (x_i + D))] & \text{for } \xi^M. \end{cases} \tag{2}$$

In both cases, the averaged perturbed state is equal to the unperturbed state $\langle x_i' \rangle = \int_I x_i' f(x_i', x_i) dx_i' / \int_I dx_i' = x_i$, reflecting symmetric perturbations.

The decay of the unstable fixed point at $x=0$ is quantified by the *residence time (escape time)* $T(x_0, D)$, representing the mean time (with respect to different noise realizations) a trajectory with initial state x_0 spends on I. The *mean residence time* $T(D)$ with respect to the initial states is introduced as

$$T(D) = \frac{\int_I T(x_0, D) dx_0}{\int_I dx_0}. \tag{3}$$

III. DYNAMICAL PERTURBATIONS OF AN UNSTABLE FIXED POINT

The decay of an unstable state is discussed for a linear map F on the interval $I = [-1,1]$

$$F(x_i) = a x_i, \quad a > 1, \tag{4}$$

with an unstable fixed point at the origin $x=0$. For a given initial state x_0, the number of iterations to reach the boundary of I is given by the residence time

$$T(x_0, D=0) = -\frac{\ln x_0}{\ln a} \tag{5}$$

and the mean residence time

$$T(D=0) = \langle T(x_0, D) \rangle_I = 1/\ln a. \tag{6}$$

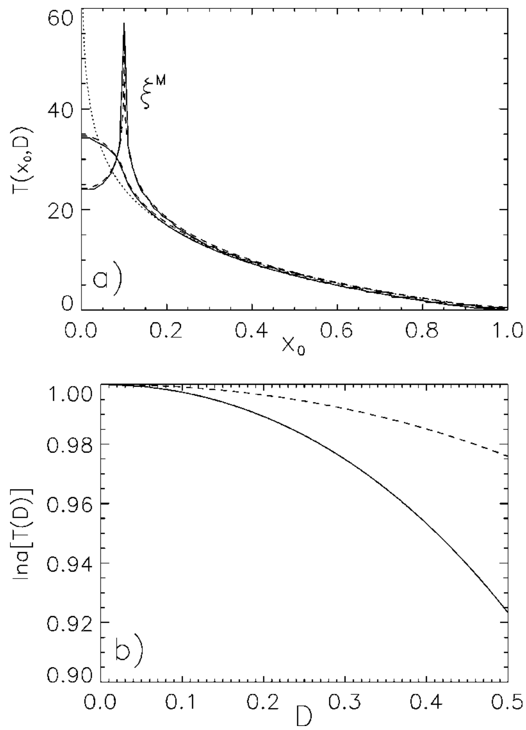

FIG. 1. (a) Dependence of the residence time $T(x_0, D)$ on the initial state x_0 for the linear map ($a=1.1$, $D=0.1$) with dichotomous Markov noise ξ^M and uniform noise (full line: analytical result; dashed line: numerical simulation). The dotted line marks the noiseless residence time. (b) The average residence time $T(D)$, scaled with the slope a, as a function of the noise amplitude D.

In the presence of dynamical noise the escape time $T(x_0, D)$ is simulated for a given number of noise realizations, and the mean escape time $T(D) = \langle T(x_0, D) \rangle$ is determined for a set of 100 initial values, which are equidistantly distributed on the interval $[0,1]$. In certain cases analytical expressions can be derived and compared with numerical simulations.

A. Perturbation of initial states

Trajectories to given initial values x_0 are perturbed at the initial time step $i=0$, and continue noiselessly for $i>0$ [$g(x_i, \xi_i) = \xi_i \delta_{i,0}$ in Eq. (1)]. Then the residence time is given by

$$T(x_0, D) = \frac{\int_I T(x_0', D=0) f(x_0', x_0) dx_0'}{\int_I dx_0'}. \tag{7}$$

For dichotomous Markov noise this results in

$$T(x_0, D) = \begin{cases} -\dfrac{\ln|D^2 - x_0^2|}{2\ln a} & \text{for } |x_0| < 1 - D \\ -\dfrac{\ln(x_0 - D)}{2\ln a} & \text{for } |x_0| \geqslant 1 - D, \end{cases} \tag{8}$$

and for uniform noise in

$$T(x_0,D)=\begin{cases} -\dfrac{(x_0+D)\ln(x_0+D)+(D-x_0)\ln|D-x_0|-2D}{2D\ln a} & \text{for } |x_0|<1-D \\ -\dfrac{(D-x_0)\ln|D-x_0|+x_0-D-1}{2D\ln a} & \text{for } |x_0|\geqslant 1-D. \end{cases} \tag{9}$$

In Fig. 1(a) it is shown that simulated and analytical residence times coincide rather well. For both random processes $\xi^{\rm uni}$ and ξ^M, certain initial values x_0 exist which are characterized by an increased residence time, in comparison to the noiseless one. The intersection point of the graphs $T(x_0,D=0)$ and $T(x_0,D)$ does not depend on the slope, but depends on the noise level D. For dichotomous Markov noise this intersection point can be calculated analytically as $D/\sqrt{2}$. A further result is that the corresponding amplification factor $T(x_0,D)/T(x_0,D=0)$ is independent of the slope a, whereas the difference $T(x_0,D)-T(x_0,D=0)$ grows with decreasing slope as $1/\ln a$. However, despite the noise-induced enlargement of the residence time for certain initial values, the average residence time over all initial values

$$T(D)=\langle T(x_0,D)\rangle_I=\begin{cases} -\dfrac{(1-D)\ln(1-D)+D-2}{2\ln a} & \text{for } \xi^M \\ -\dfrac{1.5D^2-5D-(1-D)^2\ln(1-D)}{4D\ln a} & \text{for } \xi^{\rm uni} \end{cases} \tag{10}$$

decreases with the noise level for dichotomous Markov noise as well as for uniform white noise [Fig. 1(b)]. Otherwise, perturbing only initial states, which are larger than the intersection point of the noisy and noiseless graph, e.g., $x_0>D/\sqrt{2}$ for dichotomous Markov noise, a noise-induced increase of the mean residence time can be generated. Nevertheless, an asymmetry between the effect of a positive and a negative noise event can give rise to stabilizing properties, although the distribution of noise is symmetric.

B. Dynamical perturbations on a small interval: $[\alpha,a\alpha]$

In contrast to the above case, the decay of an unstable state is discussed when dynamical noise is restricted to a small subinterval $[\alpha,\beta]\in I$ and $[g(x_i,\xi_i)=\xi_i\Theta(x_i-\alpha)\Theta(\beta-x_i)$ in Eq. (1)]. For $\beta=a\alpha$ this subinterval is large enough such that any (un)perturbed trajectory with $0<x_0<\beta$ enters it at least once, but also small enough such that a trajectory is perturbed only once for $D\xi_i>\beta-\alpha$, each time it enters this subinterval. (In the following, only this case $D\xi_i>\beta-\alpha$ is discussed.) As a consequence of the fact that a negative or positive noise event can throw a trajectory $(x_0<\beta)$ out from the subinterval but not back into it, an increase of the average number of iterations on I is possible for $|a|>1$. In a detailed analysis, analytical expressions for the mean residence time $T(D),T(D)>T(D=0)$, are derived, as well as conditions for an optimized enhanced stability of the unstable fixed point by a proper choice of α.

The residence time $T(x_0,D)$ contains a noiseless dynamics on $[0,\alpha]$ and on $[\beta,1]$, superimposed by the effect of perturbations on $[\alpha,\beta]$. Thus for any $x_0\in I$ the residence time can be described exactly, whereby only the effect of positive and negative perturbations in $[\alpha,\beta]$ is averaged:

$$T(x_0,D)=(T(x_0,D=0)-T(\alpha,D=0))\Theta(\alpha-x_0)+(T^-_{[\alpha,\beta]}(D)-T_{[\alpha,\beta]}(D=0)+T^+_{[\alpha,\beta]}(D))\Theta(\beta-x_0) + T(x_0,D=0)\Theta(x_0-\beta), \tag{11}$$

where $T^-_{[\alpha,\beta]}(D)=\langle T^-(x_0,D)\rangle_{[\alpha,\beta]}$, $T^+_{[\alpha,\beta]}(D)=\langle T^+(x_0,D)\rangle_{[\alpha,\beta]}$, or $T_{[\alpha,\beta]}(D=0)=\langle T(x_0,D=0)\rangle_{[\alpha,\beta]}$, respectively, represents the mean residence time for trajectories starting in $[\alpha,\beta]$ and getting perturbed by a negative, positive, or no noise event. For analytical expressions, see the Appendix. Therefore, the mean enlargement of the residence time by a negative noise event is given by the difference $T^-_{[\alpha,\beta]}(D)-T_{[\alpha,\beta]}(D=0)$. Further, it is already taken into account in Eq. (11) that any trajectory $(x_0<\beta)$ can be perturbed once by a positive noise event, but several times by a negative noise event. Nevertheless, it can be shown that on average a trajectory is perturbed also once by a negative noise event [17].

In Fig. 2 it is illustrated that dynamical perturbations on $[\alpha,\beta]$ lead to an enlargement of $T(x_0,D)$ in $[0,\beta]$, for both random processes. This effect is clearly reduced for uniform noise in comparison to ξ^M for this special set of parameters. Nevertheless, in both cases, the stabilizing phenomenon remains after averaging over all initial states, in contrast to the previous case in Sec. III A.

In the next step, the parameters α and D are discussed in order to optimize the noise-induced stabilizing effect. For this analysis Eq. (11) can be rewritten as (compare Fig. 2)

$$T(x_0,D)=T(x_0,D=0)+\Delta T_{[\alpha,\beta]}(D)\Theta(\beta-x_0), \tag{12}$$

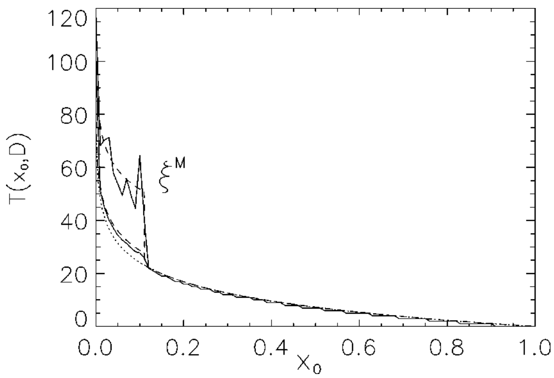

FIG. 2. Residence time $T(x_0,D)$ vs initial states x_0 for the linear map ($a=1.1, D=0.1, \alpha=0.1$) with dichotomous Markov noise ξ^M and uniform noise (full line: analytical result; dashed line: numerical simulation; dotted line: noiseless case). The peak at $x_0=D$ in the numerical simulation does not appear in the analytical graph, since $T(x_0,D)$ is an average quantity on $[\alpha,\beta]$. For uniform noise the analytical residence time is slightly underestimated, since noise events $D\xi_i<\beta-\alpha$ exist for any D.

where $\Delta T_{[\alpha,\beta]}(D)$ describes the mean net enlargement of the noiseless residence time

$$\Delta T_{[\alpha,\beta]}(D)=T^{-}_{[\alpha,\beta]}(D)+T^{+}_{[\alpha,\beta]}(D)-2T_{[\alpha,\beta]}(D=0). \tag{13}$$

The optimal interval $[\alpha,\beta=a\alpha]$, e.g., the optimal α, for a given noise amplitude D, for which $\Delta T_{[\alpha,\beta]}(D)$ takes its maximum value is approximated by the coincidence of $(\alpha+\beta)/2$ with the state x_c, for which $\Delta T(x_0,D)=T^{-}(x_0,D)+T^{+}(x_0,D)-2T(x_0,D=0)$ is maximal. This symmetric choice of the optimal interval, denoted as $[\alpha_c,\beta_c=a\alpha_c]$, around the critical state x_c is a good approximation, since the subinterval is small, at least for weakly unstable fixed points, although $\Delta T(x_0,D)$ is not exactly symmetric with respect to x_c. In the case of dichotomous Markov noise, $\Delta T(x_0,D)$ is maximal for $x_c=D$, yielding $\alpha_c=2D/(1+a)$. For uniform noise, $\Delta T(x_0,D)$ is maximal for $x_c=\eta D$, where $\eta=0.834$ is calculated numerically. In Fig. 3 the dependence of $T_{[\alpha,\beta]}(D)$ on α confirms that α_c is a good approximation. Even for uniform noise, where α_c is shifted slightly from the exact maximum, the maximum mean residence time can be determined within an error of 5% when using α_c.

The mean net enlargement $\Delta T_{[\alpha_c,\beta_c]}(D)$ of the residence time in Eq. (12) does not depend on the noise level D, as long as the critical boundary α_c is concerned. However, it depends very strongly on the slope a of the map, such that it causes remarkable stabilization only for fixed points which are weakly unstable. For $a=1.1$ (1.01) and dichotomous Markov noise, one obtains $\Delta T_{[\alpha_c,\beta_c]}(D)=35.2$ (563.8), where it is $\Delta T_{[\alpha_c,\beta_c]}(D)=8.6$ (82.7) for uniform noise.

Finally the optimized [18] mean residence time, calculated from Eq. (12), is given by

$$T(D)=T(D=0)+\Delta T_{[\alpha_c,\beta_c]}(D)\beta_c, \tag{14}$$

which leads to

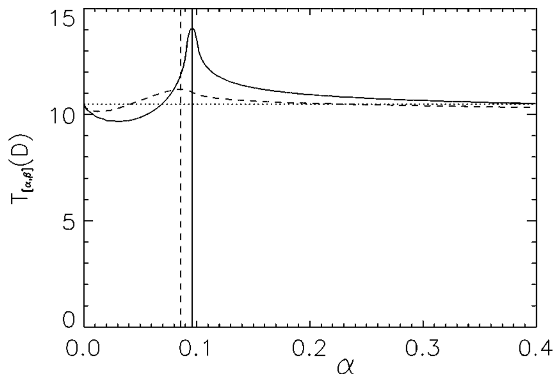

FIG. 3. Mean residence time $T_{[\alpha,\beta]}(D)$ vs α for the linear map ($a=1.1, D=0.1$). For dichotomous Markov noise (full line) the maximum appears at $\alpha=0.096$, in comparison to $\alpha_c=0.095$. For uniform noise (dashed line) the maximum appears at $\alpha=0.086$, whereas $\alpha_c=0.079$.

$$T(D)=T(D=0)+\Delta T_{[\alpha_c,\beta_c]}(D)\frac{2a}{1+a}D \tag{15}$$

for dichotomous Markov noise. Thus the optimized residence time $T(D)$ increases linearly with D, and depends strongly on the slope a. For $a=1.1$, $T(D)=10.5+36.9D$, whereas $T(D)=100.5+566.6D$ for $a=1.01$. That is, dynamical noise on a small interval can clearly induce stabilizing effects on the decay of an unstable state, even by averaging over all initial states.

C. Dynamical perturbations on a larger interval: $[\alpha,1]$

Dynamical noise, acting locally in state space, can enlarge the mean residence time, $T(D)>T(0)$. This stabilizing effect can be drastically amplified, if the length of the subinterval is increased to $[\alpha_c,1]$. Then a diffusion process is superimposed on the discussed stabilization phenomenon, such that states $x_i>a\alpha_c$ can reenter the subinterval $[\alpha_c,a\alpha_c]$ and again contribute to the enlargement of $T(D)$. Numerical simulations demonstrate that the increase of $T(D)$ for dynamical noise on the interval $[\alpha,1]$ and for a given amplitude D takes its maximum value, if $\alpha=\alpha_c$. This is true for dichotomous Markov noise with an error of 1%, and for uniform noise with an error of 3–6 %, which is considered as negligible.

In Fig. 4 the residence time $T(x_0,D=0.1)$ for dichotomous Markov noise, as well as for uniform dynamical noise, is compared with the noiseless residence time. A clear enlargement of the residence time under the presence of the diffusion process on $[\alpha_c,1]$ exists in both cases. In the following, the details are discussed only for dichotomous Markov noise, since it is easier to handle ($|\xi_i|=1=\text{const}$), but can be transferred directly to the case of uniform noise, where the enlargement is reduced because of $\langle|\xi^M|\rangle=2\langle|\xi^{\text{uni}}|\rangle$.

This enlargement of the decay time of the unstable fixed point still holds for the average quantity $T(D)=\langle T(x_0,D)\rangle$, $T(D)>T(D=0)$, and is clearly enhanced in comparison to the case, where dynamical noise is restricted to the subinterval $[\alpha_c,a\alpha_c]$ as seen in Fig. 5. To investigate the influence of the diffusion mechanism on this reduction of

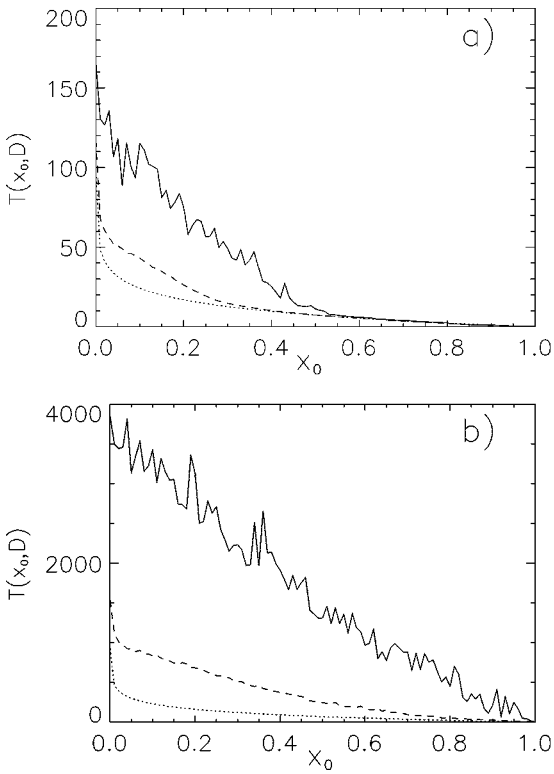

FIG. 4. The residence time $T(x_0,D)$ vs the initial state x_0 for a given noise amplitude $D=0.1$ and different slopes (a) $a=1.1$ and (b) $a=1.01$: dichotomous Markov noise (full line), uniform noise (dashed line), and the noiseless case (dotted line).

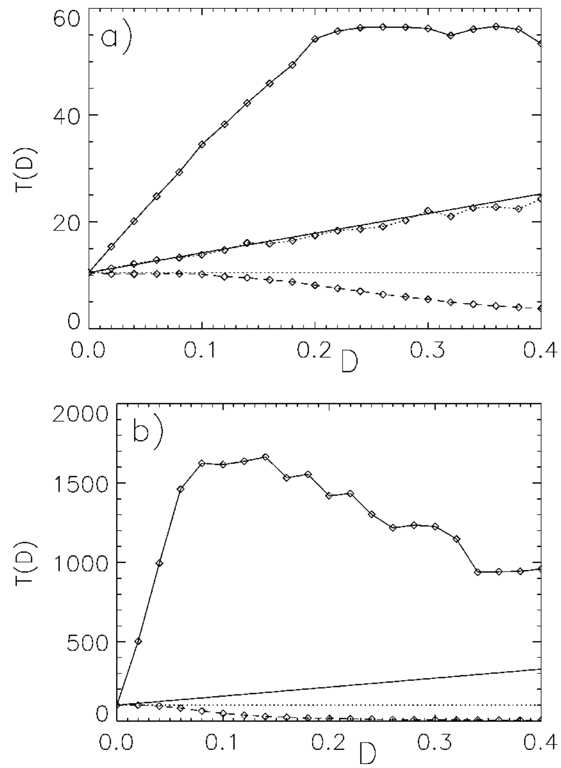

FIG. 5. Mean residence time $T(D)$ vs the noise amplitude D for different slopes (a) $a=1.1$ and (b) $a=1.01$: dynamical noise ξ^M on $[\alpha_c,1]$ (full line with ◇, 200 noise realizations), dynamical noise ξ^M on $[\alpha_c,a\alpha_c]$ (dotted line with ◇, 200 noise realizations; full line, analytical result [Eq. (15)]), and global dynamical noise (dashed line with ◇, 200 noise realizations). The dotted lines mark the noiseless residence time $T(D=0)$.

instability separately, the amplifying factor Φ is defined as the fraction of the residence time $T(D)$ with dynamical perturbations on $[\alpha_c,1]$, and the corresponding residence time with dynamical perturbations on $[\alpha_c,a\alpha_c]$, only. As presented in Fig. 6(a), the typical shape of the curve of Φ versus D does not dependent on the slope a. It is characterized by a maximum value of Φ at an intermediate noise level $D_{\max}$.

To understand the influence of the diffusion process on $T(D)$, it is pointed out that dynamical noise shifts the graph of the map F [Eq. (4)] by $\pm D$. This corresponds to a shift of the unstable fixed point to $x^*=\pm aD/(a-1)$ for $\mp D$. (Again, because of the antisymmetric property of the map, only the dynamics on $[0,1]$ is discussed.) As a consequence of the shifted fixed point, states x_i with $a\alpha_c<x_i<x^*$ can undergo a backward diffusion (by negative noise events), reenter the subinterval, where no dynamical noise acts, and thus further enlarge $T(D)$.

The typical D dependence of the amplifying factor Φ is determined by a competition between two properties of a trajectory, with initial state x_0, to reach the boundary at $x=1$ and to reach the boundary at $x=\alpha_c$, where both properties become more likely with increasing D. For simplicity only the extreme cases are discussed, e.g., a trajectory is perturbed by negative (positive) perturbations only. For this, n^- is defined as the minimum number of iterations, such that a trajectory, starting at $x_0=1$, reaches a state $x_{n^-}<\alpha_c$ by negative perturbations $\xi_i=-D$, and correspondingly, n^+ is defined as the minimum number of iterations such that a trajectory, starting at $x_0=\alpha_c$, reaches a state $x_{n^+}>1$ by positive perturbations $\xi_i=+D$. The difference $n^- - n^+$ is plotted in Fig. 6(b). For $D>D_{\max}$, where $D_{\max}=0.2$ for $a=1.1$, both minimum numbers are similar, $n^-\approx n^+$, reflecting a symmetry between the effect of a negative and a positive noise event on $[\alpha_c,1]$. Therefore, the decrease of the amplifying factor Φ with increasing D ($D>D_{\max}$) is completely described by a diffusion process with increasing diffusion constant. In contrast, for $D<D_{\max}$, an asymmetry between n^- and n^+, where $n^-\gg n^+$ for small D, characterizes the diffusion process, although the noise distribution is symmetric. The combination of the fact that the probability for finding n^- successive negative perturbations in a finite sequence of noise realizations is rather small, together with the fact that n^- decreases with increasing noise D, dominates the competition by an increase of Φ with D. That is, for $D=D_{\max}$ there already exist subsequences in the noise realization, such that a trajectory with initial state $x_0=1$ can reach a state $x_i<\alpha_c$, whereas for smaller amplitudes $D<D_{\max}$ only states $x_0<1$ are allowed to reach the boundary at α_c. In addition to this, the asymmetry between n^- and n^+ becomes smaller, when the shifted fixed point x^* leaves the interval I, which happens for $a=1.1$ at $D=0.09$ $[x^*(D=0.09)=1]$. At this noise level the amplifying factor Φ changes its slope, as seen in Fig. 6(a). Further it is briefly mentioned that a comparison of different unstable states tells that $D_{\max}$ increases with the slope a, which is a consequence of an asymmetry between n^- and n^+ for even larger noise amplitudes.

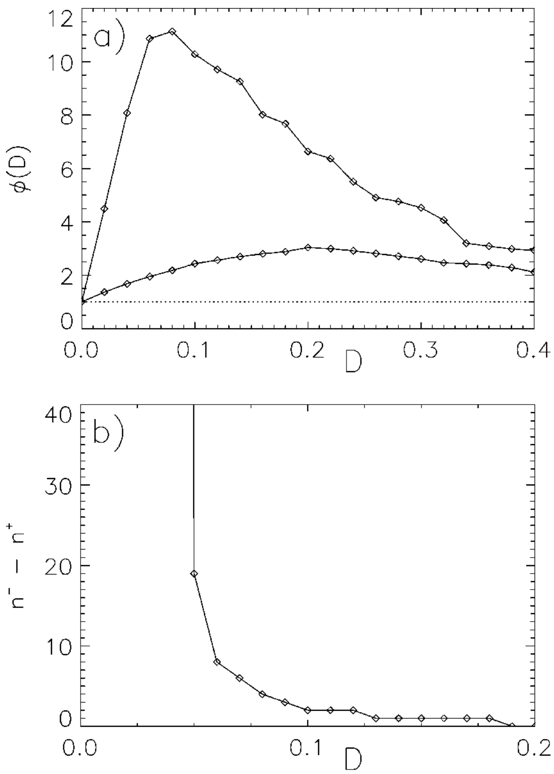

FIG. 6. (a) Amplifying factor Φ vs D for different slopes, $a=1.01$ (above curve) and $a=1.1$. The dotted line marks $\Phi=1$. (b) The asymmetry $n^- - n^+$ between backward and forward iterations on the interval $[\alpha_c,1]$ as a function of the noise level D for a map with slope $a=1.1$ and dichotomous Markov noise.

D. Global dynamical perturbations

Concerning dynamical perturbations on the entire interval I [$g(x_i,\xi_i)=\xi_i$ in Eq. (1)], the noisy dynamics is characterized completely by the shifted fixed point x^*, and no longer by the fixed point at $x=0$. Therefore, the mechanism causing the enlargement of the residence time near the unstable fixed point $x=0$ (Sec. III B) is no longer active. As expected, one finds a decrease of the mean residence time $T(D)$ with the noise level D [Figs. 5(a) and 5(b)]. Nevertheless, initial states $x_0, x_0<x^*$, exist, such that dynamical noise increases $T(x_0,D)$ in comparison to the unnoisy case (Fig. 7). This is a consequence of the shifted fixed point x^*, allowing backward diffusion, which is not present for $D=0$.

Since global dynamical noise does not enhance the mean residence time $T(D)$, it is only addressed briefly why $T(D)$ remains constant for small noise levels D. As presented in Figs. 5(a) and 5(b), $T(D)$ starts to decrease with D, when the critical noise level, for which x^* leaves I, is exceeded. For $a=1.1$ this happens for $D=0.09$ and for $a=1.01$ for $D=0.01$. In contrast to the previous case (Sec. III C), the time scale of the dynamics is determined only by the number of iterations a trajectory with $x_0=0$ needs to reach $x_{n^+}=x^*$. Because $n^+=\ln 2/\ln a$ is independent of D, n^+ starts to decrease as soon as $x^*>1$, which is the onset of the decrease of $T(D)$.

If one introduces an asymmetry of the noise-induced shifted fixed points [$x^*=\pm aD/(a-1)$] by a change of the slope at $x=0$ such that $a=a_1$ for $x<0$ and $a=a_2, a_2>a_1$ for $x\geqslant 0$ in Eq. (4), global dynamical noise yields a slight increase of the mean residence time $T(D)=\langle T(x_0,D)\rangle_{[0,1]}$ when averaging over the initial states in [0,1]. This is consistent with the results proposed by Agudov [5] in the context of continuous dynamical systems. However, for initial values in $[-1,0]$, the corresponding averaged residence time $T(D)=\langle T(x_0,D)\rangle_{[-1,0]}$ decreases with D, dominates the stabilizing contribution of [0,1], and even yields a decrease of the mean residence time $T(D)$ with the noise amplitude on the entire interval $I=[-1,1]$.

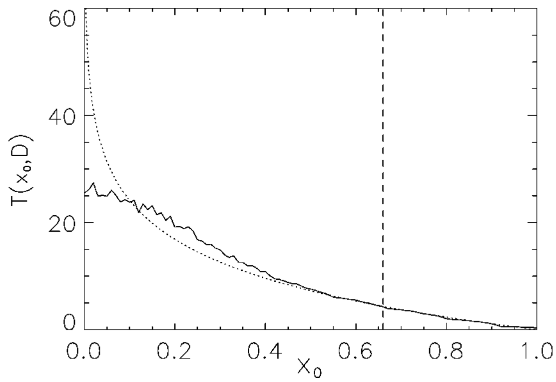

FIG. 7. Mean residence time $T(x_0,D)$ for global dynamical noise with amplitude $D=0.06$ (full line) and for the noiseless case (dotted line). The slope of the map is $a=1.1$. The dashed line marks the noise-induced shifted fixed point x^*.

IV. APPLICATION: DETERMINISTIC DIFFUSION

The study of a periodic continuation of the map in Eq. (4) is physically motivated, such that it exhibits deterministic diffusive behavior. It is well accepted that essential properties of deterministic diffusion are already contained in simple one-dimensional periodic maps [7,8]. Based on previous results for noise-induced delay of unstable states, the conjecture is that dynamical noise can decrease the deterministic diffusion under certain conditions.

Consider the linear map F in Eq. (4), which is continued periodically beyond the interval $[-1,1]$ onto the real line by a lift of size 2, such that $F(x+2)=F(x)+2$ and F antisymmetric [Fig. 8(a)]. In the context of deterministic diffusion, x_0 is called the injection point on $[-1,1]$. To obtain a homogeneous distribution of injection points, F is chosen to be a continuous, piecewise linear map with the requirements $F(1/a)=1$ and $F(1)=3$. The shifted noisy maps are continued in an analogous way, as illustrated in Fig. 8(a), such that all graphs intersect at $F(1)=3$. To be comparable with Sec. III, the number of iterations on $[-1,1]$ is regarded as residence time of the deterministic diffusion process.

As expected, global dynamical noise ξ^M yields a decrease of the mean residence time $T(D)$ for the deterministic diffusion process, which coincides with the corresponding $T(D)$ in Sec. III D, since by construction of the continuation of F in Fig. 8(a) the distribution of injection points is homogeneous.

However, dynamical perturbations on $[\alpha_c,1]$ (and, by symmetry, also on $[-1,-\alpha_c]$) yield a delay of the deterministic diffusion process as illustrated in Fig. 8(b). For this process, on average, the mean residence time $T(D)$ is the

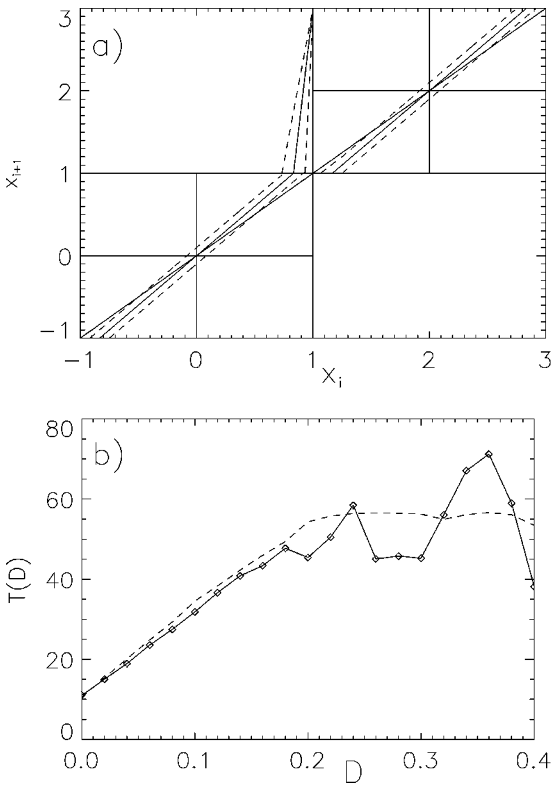

FIG. 8. (a) Periodic continuation of the linear map in Eq. (4) (full line). The dashed lines, above and below this curve, correspond to the noisy maps ($a=1.1$, $D=0.08$). (b) Corresponding residence time $T(D)$ in $[-1,1]$ with dynamical perturbations on $[\alpha_c,1]$ (full line, 10^6 iterations). The dashed line represents the residence time of the single map, discussed in Sec. III C.

same as for the isolated map in Sec. III C, but with large fluctuations superposed on it. These fluctuations are not caused by low statistics, since they are robust against the increase of the number of iterations (from 10^6 to 10^7). Moreover, they can be understood as a consequence of an inhomogeneous distribution of injection points on $[-1,1]$. The local restriction of the perturbations, and therefore the frequent reentering of the trajectory into the unperturbed region $x<\alpha_c$ by backward iteration (as discussed in Sec. III C), yields an increased frequency of states x_i near α_c, which evolves by positive perturbations $+D$ to the cell boundary at $x=1$. Thus negative fluctuations [minima in $T(D)$] are expected, if α_c is mapped to injection points near $x=1$. This happens for noise levels D for which the nth iterate is determined by $F^n(\alpha_c+D)=1$ and $\alpha_c=2D/(1+a)$. For $n=1$, one obtains $D=(1+a)/(3a+a^2)=0.47$ for $a=1.1$; analogously, $n=2$ corresponds to $D=0.29$, and $n=3$ to $D=0.2$. These critical noise levels fit well with the minima of $T(D)$ in Fig. 8(b). Correspondingly, the maxima in $T(D)$ appear at noise levels, for which α_c is mapped onto injection points near $x=0$ by positive perturbations. After some calculation, one finds $D=0.36$ for the first maximum in $T(D)$.

For smaller noise levels, $T(D)$ depends rather linearly on D, and the fluctuations disappear, since the number of iterations of the state α_c on $[\alpha_c,1]$ increases. In particular, for $D<0.16$, the number of iterations is already large enough such that dynamical noise can spread the initial inhomogeneity of states near α_c over the entire interval I.

V. CONCLUSIONS

Dynamical noise (dichotomous Markov noise, uniform white noise, and also Gaussian white noise [19]) can induce a clear enhancement of the stability of an unstable fixed point, in particular for weakly unstable states, and a clear enhancement of localization in deterministic diffusion models. In both cases, even the mean residence time of a typical trajectory near the unstable state increases with the noise level, if noise is added locally in state space, although the distribution of the noise events is symmetric. This stabilizing property is caused by a combination of two facts: (1) Since noise can throw a trajectory out from a subinterval, but not back into it, the number of iterations on the entire interval increases, if the fixed point is unstable. (2) Since noise shifts the unstable fixed point, backward iteration is possible on the interval. By numerical and analytical investigations, parameters can be derived for which the stabilizing phenomenon is optimized for a given random process.

Of course this noise-induced enhancement of stability does not represent a true stabilization in the sense of controlling. Nevertheless, it is interesting to know to what extent a (simple type of multiplicative) dynamical noise can generate stabilization or localization in deterministic dynamical systems. On the other side, since noise is ubiquitous in natural and experimental systems, the (measured) time scale of the decay of an unstable state can be essentially changed by random interactions of the deterministic system.

The presented analysis and the corresponding stabilizing mechanisms are not restricted to unstable fixed points, but still hold for unstable periodic orbits of larger period in (non)linear dynamical systems. Also, a generalization to intermittent behavior is straightforward, which is related further to diffusion in Hamiltonian systems as well as (for example) to turbulence in dissipative dynamical systems [12]. There, strange kinetics as trapping and flights of particles is present, which can be discussed successfully within the concept of Lévi processes.

In further studies the possible influence of these results on conductivity in general, and on thermodynamical relations, should be addressed. Since unstable periodic orbits or intermittent behavior are present in many physical, chemical, or biological systems, experimental realizations of noise-induced stabilization and noise-induced localization should be possible.

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APPENDIX

By symmetry of the linear map in Eq. (4), only initial states $x_0\in[0,1]$ are considered. The residence time for the unperturbed map is given by

$$T(x_0,D=0)=-\frac{\ln x_0}{\ln a}, \tag{A1}$$

yielding the mean noiseless residence time on $[\alpha,\beta]$

$$T_{[\alpha,\beta]}(D=0)=\frac{\int_\alpha^\beta T(x_0,D=0)dx_0}{\beta-\alpha}=-\frac{\beta\ln\beta-\alpha\ln\alpha+\alpha-\beta}{(\beta-\alpha)\ln a}. \tag{A2}$$

For dichotomous Markov noise the residence time for negative $(-)$ or positive $(+)$ perturbations is represented by

$$T^{\mp}(x_0,D)=\int_I T(x_0',D=0)\,\delta(x_0'-(x_0\mp D))dx_0'=-\frac{\ln|x_0\mp D|}{\ln a}, \tag{A3}$$

which corresponds to the mean residence time on $[\alpha,\beta]$:

$$T^{\mp}_{[\alpha,\beta]}(D)=\langle T^{\mp}(x_0,D)\rangle_{[\alpha,\beta]}=-\frac{(\beta\mp D)\ln|\beta\mp D|-(\alpha\mp D)\ln|\alpha\mp D|+\alpha-\beta}{(\beta-\alpha)\ln a}. \tag{A4}$$

For uniform noise, the residence time for negative $(-)$ or positive $(+)$ perturbations is given by

$$T^{-}(x_0,D)=\int_I T(x_0',D=0)\frac{\Theta(x_0'-(x_0-D))\Theta(x_0-x_0')}{D}dx_0'=-\frac{(D-x_0)\ln|D-x_0|+x_0\ln x_0-D}{D\ln a}, \tag{A5}$$

$$T^{+}(x_0,D)=\int_I T(x_0',D=0)\frac{\Theta(x_0'-x_0)\Theta(x_0+D-x_0')}{D}dx_0'=-\frac{(D+x_0)\ln|D+x_0|-x_0\ln x_0-D}{D\ln a}, \tag{A6}$$

which leads to the corresponding mean residence time on $[\alpha,\beta]$

$$T^{\mp}_{[\alpha,\beta]}(D)=\langle T^{\mp}(x_0,D)\rangle_{[\alpha,\beta]}=-\frac{\pm\beta^2\ln\beta\mp\alpha^2\ln\alpha\mp(\beta\mp D)^2\ln|\beta\mp D|\pm(\alpha\mp D)^2\ln|\alpha\mp D|-3D(\beta-\alpha)}{2D(\beta-\alpha)\ln a}. \tag{A7}$$

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[17] Let $g:=T^{-}_{[\alpha,\beta]}(D)-T_{[\alpha,\beta]}(D=0)$ be the average gain of iteration steps by a single negative noise event acting in $[\alpha,\beta]$: The probability for 1 $(2,\ldots)$ negative perturbation is $\frac{1}{4}$ $(\frac{1}{8},\ldots)$, since it is caused by a noise sequence $-+(--+,\ldots)$. Thus, the total gain of iterations by negative noise events is given by $g/4+(2g)/8+(3g)/16\cdots=g\Sigma_{n=1}^{\infty}n/2^{n+1}=g$, e.g., it is just the gain of iterations for one single negative perturbation.

[18] To be exact, for maximizing the mean residence time $T(D)$, the product of $\Delta T_{[\alpha,\beta]}(D)$ and $\beta=a\alpha$ should be optimized with respect to α. However, a usually strong decrease of $\Delta T_{[\alpha,\beta]}(D)$ near $\alpha=\alpha_c$ dominates the optimization, which is why α_c is still a good approximation.

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